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A matrix sequence $\{\Gamma(A^m)\}_{m=1}^{\infty}$ might converge even if the matrix A is not primitive

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ABSTRACT

It is well-known that, for an irreducible Boolean $(0, 1)$-matrix A, the matrix sequence $\{A^m\}_{m=1}^{\infty}$ converges if and only if A is primitive. In this paper, we introduce an operation Γ on the set of Boolean $(0, 1)$-matrices such that a matrix sequence $\{\Gamma(A^m)\}_{m=1}^{\infty}$ might converge even if the matrix A is not primitive. Given a Boolean $(0, 1)$-matrix A, we define a matrix $\Gamma(A)$ so that the (i, j)-entry of $\Gamma(A)$ equals 0 if for $i \neq j$, the inner product of the ith row and jth row of A is 0 and equals 1 otherwise.

The aim of this paper is to study the convergence of $\{\Gamma(A^m)\}_{m=1}^{\infty}$ for a Boolean $(0, 1)$-matrix A whose digraph has at most two strong components. We show that $\{\Gamma(A^m)\}_{m=1}^{\infty}$ converges to a very special type of matrix as m increases if A is an irreducible Boolean matrix. Furthermore, we completely characterize a Boolean $(0, 1)$-matrix A whose digraph has exactly two strongly connected components and for which $\{\Gamma(A^m)\}_{m=1}^{\infty}$ converges, and find the limit of $\{\Gamma(A^m)\}_{m=1}^{\infty}$ in terms of its digraph when it converges. We derive these results in terms of the competition graph of the digraph of A.

1. Introduction

The focus of this paper is a problem about the convergence of a certain sequence of Boolean $(0, 1)$-matrices or equivalently the convergence of the m-step competition graphs of certain digraphs.

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$$A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad A^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad A^3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \quad A^4 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \cdots$$

$$A' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Fig. 1. Note that A, A^2, A^3 are all distinct and $A^4 = A$. Thus $\{A^m\}_{m=1}^{\infty}$ does not converge. However $\Gamma(A^m) = A'$ for each positive integer m.

Throughout the paper, we will state definitions, facts, and theorems (where appropriate) in terms of both Boolean (0, 1)-matrices and competition graphs.

For the two-element Boolean algebra $\mathcal{B} = \{0, 1\}$, $\mathcal{B}_n$ denotes the set of all $n \times n$ matrices over $\mathcal{B}$. Under the Boolean operations, we can define matrix addition and multiplication in $\mathcal{B}_n$. Given a matrix A in $\mathcal{B}_n$, we define a matrix $\Gamma(A) = (\gamma_{ij}) \in \mathcal{B}_n$ by

$$\gamma_{ij} = \begin{cases} 0 & \text{if } i = j; \\ 0 & \text{if } i \neq j \text{ and the inner product of row } i \text{ and row } j \text{ of } A \text{ is } 0; \\ 1 & \text{if } i \neq j \text{ and the inner product of row } i \text{ and row } j \text{ of } A \text{ is not } 0. \end{cases}$$

As a matter of fact, for a matrix $A \in \mathcal{B}_n$, $\Gamma(A)$ is the adjacency matrix of the competition graph of the digraph of A. Given a matrix A in $\mathcal{B}_n$, there exists a unique digraph whose adjacency matrix is A. We call such a digraph the *digraph of* A and denote it by $D(A)$.

Given a digraph D, the *competition graph* $C(D)$ of D has the same vertex set as D and has an edge between vertices u and v if and only if there exists a common prey of u and v in D. If (u, v) is an arc of a digraph D, then we call v a *prey* of u (in D) and call u a *predator* of v (in D). A graph G is called the *row graph* of a matrix M if the rows of M are the vertices of G, and two vertices are adjacent in G if and only if their corresponding rows have a nonzero entry in the same column of M. This notion was studied by Greenberg et al. [6]. As noted in [6], the competition graph of a digraph D is the row graph of its adjacency matrix. Thus it can easily be checked that the adjacency matrix of the competition graph of a digraph D is $\Gamma(A)$ where A is the adjacency matrix of D.

The notion of competition graph is due to Cohen [5] and has arisen from ecology. Competition graphs also have applications in coding, radio transmission, and modeling of complex economic systems. (See [13,14] for a summary of these applications.)

The greatest common divisor of all lengths of directed cycles in a nontrivial strongly connected digraph D is called the *index of imprimitivity of* D. A digraph D is said to be *primitive* if D is strongly connected and has the index of imprimitivity 1. Let A be a matrix in $\mathcal{B}_n$. If $D(A)$ is strongly connected, then we say A is *irreducible*. We call the index of imprimitivity of $D(A)$ the *index of imprimitivity of* A, when A is irreducible. If $D(A)$ is primitive, then we say that A is *primitive*. For undefined terms in the following, the reader is referred to [2].

It is well-known that for an irreducible matrix A in $\mathcal{B}_n$, the matrix sequence $\{A^m\}_{m=1}^{\infty}$ converges if and only if A is primitive. Yet, a matrix sequence $\{\Gamma(A^m)\}_{m=1}^{\infty}$ might converge even if the matrix A is not primitive. For example, the mth power of the matrix A given in Fig. 1 does not converge as m increases since it is not primitive. However, the $\{\Gamma(A^m)\}_{m=1}^{\infty}$ converges to A' since $\Gamma(A^m) = A'$ for any positive integer m.

In this paper, we study the convergence of $\{\Gamma(A^m)\}_{m=1}^{\infty}$ for a matrix $A \in \mathcal{B}_n$ whose digraph has at most two strong components. We show that $\{\Gamma(A^m)\}_{m=1}^{\infty}$ converges as m increases for any irreducible Boolean matrix A and its limit is a block diagonal matrix each of whose blocks consists of all 1s up to conjugation by simultaneous permutation of rows and columns. From now on, we call such a matrix *J block diagonal* (for short *JBD*) matrix (where J means a matrix with all 1s). Furthermore, we completely characterize a matrix $A \in \mathcal{B}_n$ whose digraph has exactly two strongly connected components and for which $\{\Gamma(A^m)\}_{m=1}^{\infty}$ converges, and find the limit of $\{\Gamma(A^m)\}_{m=1}^{\infty}$ in terms of its digraph when it converges. We derive these facts in terms of the competition graph of the digraph of A.

Given a digraph D and a positive integer m, a vertex y is an *m-step prey* of a vertex x if and only if there exists a directed walk from x to y of length m. Given a digraph D and a positive integer m, the digraph D^m has the vertex set same as D and has an arc (u, v) if and only if v is an m-step prey of u. It is well-known that a digraph D is primitive if and only if D^m equals the digraph which has all possible arcs for any $m \geqslant N$ for some positive integer N (we call the smallest such integer N the *exponent* of D). Motivated by this, we say that a graph sequence $\{G_n\}_{n=1}^{\infty}$ (resp. digraph sequence) *converges* if there exists an integer N such that G_n is equal to G_N for any $n \geqslant N$. In this case, we call the graph G_N the *limit* of the graph sequence (resp. digraph sequence). Then the goals of this paper proposed above are translated into competition graph version as follows. We show that $\{C(D^m)\}_{m=1}^{\infty}$ converges to a graph with only complete components as m increases if D is strongly connected, completely characterize a digraph D with exactly two strong components for which $\{C(D^m)\}_{m=1}^{\infty}$ converges, and find the limit of $\{C(D^m)\}_{m=1}^{\infty}$ when $\{C(D^m)\}_{m=1}^{\infty}$ converges.

Given a positive integer m, the *m-step competition graph* of a digraph D, denoted by $C^m(D)$, has the same vertex set as D and has an edge between vertices u and v if and only if there exists an m-step common prey of u and v. The notion of m-step competition graph is introduced by Cho et al. [4] and one of the important variations (see the survey articles by Kim [10] and Lundgren [12] for the variations which have been defined and studied by many authors since Cohen introduced the notion of competition). Since its introduction, it has been extensively studied (see for example [1,3,7–9,11,16]). Cho and Kim [3] showed that for any digraph D and a positive integer m, $C^m(D) = C(D^m)$. Thus the limit of the graph sequence $\{C(D^m)\}_{m=1}^{\infty}$, if it exists, is the same as that of the graph sequence $\{C^m(D)\}_{m=1}^{\infty}$. Consequently studying the graph sequence $\{C(D^m)\}_{m=1}^{\infty}$ is actually studying the sequence of m-step competition graphs of D.

2. $\{\Gamma(A^m)\}_{m=1}^{\infty}$ for an irreducible matrix $A \in \mathcal{B}_n$

In this section, we study the convergence and the limit graph of $\{\Gamma(A^m)\}_{m=1}^{\infty}$ when A is an irreducible matrix in $A \in \mathcal{B}_n$. As mentioned previously, $\Gamma(A^m)$ corresponds to the competition graph of D^m where D is the digraph of A for a positive integer m.

We start with the following observation.

Theorem 2.1. *If a digraph D is trivial or each vertex of D has an out-neighbor, then $\{C(D^m)\}_{m=1}^{\infty}$ converges.*

Proof. The proposition immediately holds for a trivial digraph. Let D be a nontrivial digraph such that each vertex has an out-neighbor. To show that $E(C(D^m)) \subset E(C(D^{m+1}))$ for any integer m, take an edge uv in $E(C(D^m))$ for some positive integer m. Then there exists a vertex z in D such that (u, z) and (v, z) are arcs of D^m for some vertex z. In D, z is a common m-step prey of u and v. By the assumption on D, there exists a vertex x such that $(z, x) \in A(D)$. Then x is a common $(m + 1)$-step prey of u and v in D, that is, x is a common prey of u and v in D^{m+1}. Therefore u and v are adjacent in $C(D^{m+1})$. Thus we have shown that $E(C(D^m)) \subset E(C(D^{m+1}))$ for any integer m. Since the competition graph of a digraph is defined to be simple, $E(C(D^m)) \subset E(K_n)$ for each m where $n = |V(D)|$. Therefore, it can easily be checked that there exists an integer N such that for any $n \geqslant N$, $C(D^n) = C(D^N)$, which implies that $\{C(D^m)\}_{m=1}^{\infty}$ converges. □

Theorem 2.1 is translated into the matrix version as follows:

Corollary 2.2. *If a Boolean $(0, 1)$-matrix A has order 1 or has at least one 1 in each row, then $\{\Gamma(A^m)\}_{m=1}^{\infty}$ converges.*

It is known that if κ is the index of imprimitivity of a digraph D, then D has an ordered partition $\{U_1, U_2, \ldots, U_\kappa\}$ of $V(D)$ such that $U_{\kappa+1} = U_1$ and each arc of D issues from U_j and enters U_{j+1} for some $j = 1, 2, \ldots, \kappa$. The sets $U_1, U_2, \ldots, U_\kappa$ are called the *sets of imprimitivity of D*.

If D is a trivial digraph, then the index of imprimitivity of D is undefined. Given a strongly connected digraph D, we define $\kappa(D)$ as 1 if D is trivial and as the index of imprimitivity of D otherwise. In addition, if D is a trivial digraph, then we denote the vertex set by U_1 and call it 'the sets of imprimitivity of D'.

If a digraph D is strongly connected, then the graph sequence $\{C(D^m)\}_{m=1}^{\infty}$ converges by Theorem 2.1. Then it is natural to ask: What is the limit of the sequence? In the rest of this section, we answer the question by showing that the limit of $\{C(D^m)\}_{m=1}^{\infty}$ is the union of exactly $\kappa(D)$ complete components.

Theorem 2.3. *If D is strongly connected, then the limit of $\{C(D^m)\}_{m=1}^{\infty}$ is the disjoint union of complete graphs whose vertex sets are the sets of imprimitivity of D.*

The above theorem immediately implies the following corollary.

Corollary 2.4. *If A is an irreducible $(0, 1)$-Boolean matrix, then the limit of $\{\Gamma(A^m)\}_{m=1}^{\infty}$ is transformable into a block diagonal matrix by simultaneous permutations of their lines in which each block is in the form of J_i where J_i is a matrix with all elements 1 and i runs from 1 to the index of imprimitivity of A.*

The following is a well-known result related to the index of imprimitivity of a digraph.

Theorem 2.5 [2, Theorem 3.4.5]. *Let D be a nontrivial strongly connected digraph of order n with the index of imprimitivity κ, and A be the adjacency matrix of D. Then there exists a permutation matrix P of order n such that*

$$PA^{\ell}P^T = \begin{pmatrix} A_1 & O & \cdots & O \\ O & A_2 & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & A_r \end{pmatrix}$$

where $r = \gcd(\kappa, \ell)$ and each of $A_1, A_2, \ldots, A_r$ is an irreducible matrix with the index of imprimitivity $\frac{\ell}{r}$.

If a matrix A in $\mathcal{B}_n$ is primitive, then there exists a positive integer m such that the mth power A^m has only positive entries, and such smallest integer m is called the *exponent* of A, which is denoted by $\exp(A)$.

Suppose that D is a nontrivial strongly connected digraph. Let A be the adjacency matrix of D and $\kappa(D) = \kappa$. By Theorem 2.5, there exists a permutation matrix P such that

$$PA^{\kappa}P^T = \begin{pmatrix} A_1 & O & \cdots & O \\ O & A_2 & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & A_\kappa \end{pmatrix}, \tag{1}$$

where A_i is primitive for any $1 \leqslant i \leqslant \kappa$. Let

$$M = \kappa \cdot \max\{\exp(A_i) \mid 1 \leqslant i \leqslant \kappa\}. \tag{2}$$

For simplicity, let $E = \max\{\exp(A_i) \mid 1 \leqslant i \leqslant \kappa\}$. Take a positive integer s. Since $PA^{sM}P^T = PA^{s\kappa E}P^T = (PA^{\kappa}P^T)^{sE}$,

$$PA^{sM}P^T = \begin{pmatrix} A_1^{sE} & O & \cdots & O \\ O & A_2^{sE} & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & A_\kappa^{sE} \end{pmatrix} \tag{3}$$

by (1). Since $sE \geqslant \exp(A_i)$, any entry of A_i^{sE} is positive for each $1 \leqslant i \leqslant \kappa$. Therefore $D(A_i^{sE})$ is a digraph with all possible arcs for each $1 \leqslant i \leqslant \kappa$, and so $D(A^{sM})$ is a disjoint union of digraphs with all possible arcs, $D(A_1^{sE}), D(A_2^{sE}), \ldots, D(A_\kappa^{sE})$.

Now we obtain the following lemma:

Lemma 2.6. *Let D be a strongly connected digraph. Then there exists a positive integer M such that $C(D^{sM})$ has exactly $\kappa(D)$ components all of which are complete for each positive integer s.*

Proof. Let $\kappa = \kappa(D)$. If D is trivial, then $\kappa = 1$ and $C(D^m)$ is trivial with the sets of imprimitivity of D as the vertex for any positive integer m. Thus the lemma immediately holds for a trivial digraph. Suppose that D is not trivial and A is the adjacency matrix of D. Then by (3), for each positive integer s, D^{sM} is the disjoint union of digraphs with all possible arcs for M defined in (2). Thus $C(D^{sM})$ is the disjoint union of complete graphs whose vertex sets are $V(D_1), V(D_2), \ldots, V(D_\kappa)$, respectively. As $V(D_1), V(D_2), \ldots, V(D_\kappa)$ are the set of imprimitivity of D, we complete the proof. $\square$

Now we are ready to prove Theorem 2.3:

Proof of Theorem 2.3. By Theorem 2.1, $\{C(D^m)\}_{m=1}^{\infty}$ converges. By Lemma 2.6, there exists a positive integer M such that $C(D^{sM})$ has exactly $\kappa(D)$ components all of which are complete for each positive integer s. Thus a graph with exactly $\kappa(D)$ components all of which are complete is the limit of a subsequence $\{C(D^{sM})\}_{s=1}^{\infty}$ of $\{C(D^m)\}_{m=1}^{\infty}$ and must be the limit of $\{C(D^m)\}_{m=1}^{\infty}$. $\square$

3. $\{\Gamma(A^m)\}_{m=1}^{\infty}$ for a matrix $A \in \mathcal{B}_n$ whose digraph has exactly two strong components

In this section, we study the convergence of $\{\Gamma(A^m)\}_{m=1}^{\infty}$ for a matrix $A \in \mathcal{B}_n$ such that

$$PAP^T = \begin{bmatrix} A_1 & F \\ O & A_2 \end{bmatrix} \tag{4}$$

for a permutation matrix P of order n, a matrix F, irreducible matrices A_1 and A_2. Again, we do so by studying the convergence of $\{C(D^m)\}_{m=1}^{\infty}$ for the digraph D of A which has exactly two strong components.

If a digraph D has two strong components and its underlying graph is disconnected, then $\{C(D^m)\}_{m=1}^{\infty}$ converges and the limit is the disjoint union of complete graphs as shown in Section 2. Thus, throughout this section, we only consider a *weakly connected* digraph whose underlying graph is connected.

Throughout this section, for a digraph D with exactly two strong components, we denote the components by D_1 and D_2 so that there is no arc from a vertex in D_2 to a vertex in D_1. For $i = 1, 2$ and for the positive integer M_i defined in (2), $C(D_i^{sM_i})$ has exactly $\kappa(D_i)$ components all of which are complete for each positive integer s. We denote the sets of imprimitivity of D_i by $U_1^{(i)}, U_2^{(i)}, \ldots, U_{\kappa(D_i)}^{(i)}$.

In the rest of this section, we characterize a digraph D for which $\{C(D^m)\}_{m=1}^{\infty}$ converges, and go further to present its limit when $\{C(D^m)\}_{m=1}^{\infty}$ converges.

If both D_1 and D_2 are trivial, then it is clear that $C(D^m)$ is an edgeless graph with two vertices for any integer m. That is, if both D_1 and D_2 are trivial then $\{C(D^m)\}_{m=1}^{\infty}$ converges and the limit graph is an edgeless graph with two vertices. Thus, from now on, we only consider a digraph with exactly two strong components, at least one of which is nontrivial.

We completely characterize a digraph D with two strong components for which $\{C(D^m)\}_{m=1}^{\infty}$ converges. For a digraph D and a vertex v of D, $N_D^-(v)$ denotes the set of all in-neighbors of v.

Lemma 3.1. *Let D be a weakly connected digraph with exactly two strong components D_1 and D_2. For any two vertices $u \in U_j^{(i)}$ and $v \in U_k^{(i)}$, the length of a directed (u, v)-walk is congruent to $k - j$ modulo $\kappa(D_i)$.*

Proof. Since there is no arc from a vertex in D_2 to a vertex in D_1, a directed (u, v)-walk belongs to D_i. Since D_i is strongly connected, we may apply one of the known properties of a strongly connected digraph with the index of imprimitivity $\kappa(D_i)$ to verify the statement. $\square$

Lemma 3.2. *Let D be a weakly connected digraph with exactly two strong components D_1 and D_2. Then there exists an integer M such that $C(D^m)$ contains complete graphs whose vertex sets are $U_1^{(1)}, \ldots, U_{\kappa(D_1)}^{(1)}$, $U_1^{(2)}, \ldots, U_{\kappa(D_2)}^{(2)}$, respectively, as subgraphs for $m \geqslant M$.*

Proof. By Theorem 2.3, there exists an integer M such that $C(D_1^m)$ and $C(D_2^m)$ equal disjoint union of complete graphs whose vertex sets are $U_1^{(1)}, \ldots, U_{\kappa(D_1)}^{(1)}$, and complete graphs whose vertex sets are $U_1^{(2)}, \ldots, U_{\kappa(D_2)}^{(2)}$, respectively, for $m \geqslant M$. Since $C(D^m)$ contains $C(D_1^m)$ and $C(D_2^m)$ as subgraphs for any positive integer m, the lemma holds. $\square$

We also need the following lemma:

Lemma 3.3 [2, Lemma 3.4.3]. *Let D be a nontrivial strongly connected digraph, and $U_1, U_2, \ldots, U_{\kappa(D)}$ be the sets of imprimitivity of D. Then there exists a positive integer N such that if x and y are vertices belonging respectively to U_i and U_j, then there are directed (x, y)-walks of every length $j - i + t\kappa(D)$ with $t \geqslant N$.*

Proposition 3.4. *Let D be a weakly connected digraph with exactly two strong components D_1 and D_2 where D_1 is nontrivial and D_2 is trivial. Then $\{C(D^m)\}_{m=1}^{\infty}$ converges if and only if for the vertex v of D_2, $\left|\left\{i \mid U_i^{(1)} \cap N_D^-(v) \neq \emptyset\right\}\right| = 1$ or $\kappa(D_1)$. Moreover, the limit of $\{C(D^m)\}_{m=1}^{\infty}$ is a graph consisting of complete components if it converges.*

Proof. Since D_1 is nontrivial, by Lemma 3.3, there exists a positive integer N for which the following holds:

$$\text{If } x \text{ and } y \text{ are vertices belonging respectively to } U_i^{(1)} \text{ and } U_j^{(1)}\text{, then there are directed } (x, y)\text{-walks of every length } j - i + t\kappa(D_1) \text{ with } t \geqslant N. \quad (*)$$

We show the 'if' part of the proposition statement. By Lemma 3.2, there exists an integer M such that $C(D^m)$ contains the complete subgraphs whose vertex sets $U_1^{(1)}, \ldots, U_{\kappa(D_1)}^{(1)}, U_1^{(2)}$, respectively, as subgraphs for $m \geqslant M$. In addition, there is no edge joining a vertex in D_1 and v in $C(D^m)$ for any positive integer m since v has no out-neighbor.

Suppose that $\left|\left\{i \mid U_i^{(1)} \cap N_D^-(v) \neq \emptyset\right\}\right| = 1$. Let $\left\{i \mid U_i^{(1)} \cap N_D^-(v) \neq \emptyset\right\} = \{i^*\}$. Then for any directed walk from a vertex in $U_i^{(1)}$ to v, the term right before v on the sequence belongs to $U_{i^*}^{(1)}$ and so it has length congruent to $i^* - i + 1$ modulo $\kappa(D_1)$. Since $i^* - i + 1 \not\equiv i^* - j + 1 \pmod{\kappa(D_1)}$ if $i \not\equiv j \pmod{\kappa(D_1)}$, two vertices belonging to distinct sets of imprimitivity cannot have an m-step common prey for any positive integer m and so there is no edge joining two vertices in distinct sets of imprimitivity in $C(D^m)$ for any positive integer m. Thus $\{C(D^m)\}_{m=1}^{\infty}$ converges to the disjoint union of complete graphs whose vertex sets are $U_1^{(1)}, \ldots, U_{\kappa(D_1)}^{(1)}, \{v\}$.

Now suppose that $\left|\left\{i \mid U_i^{(1)} \cap N_D^-(v) \neq \emptyset\right\}\right| = \kappa(D_1)$. Let $u_1, \ldots, u_{\kappa(D_1)}$ be in-neighbors of v in $U_1^{(1)}, \ldots, U_{\kappa(D_1)}^{(1)}$, respectively. Take a vertex x of D_1. Then $x \in U_j^{(1)}$ for some $j \in \{1, 2, \ldots, \kappa(D_1)\}$ and, by $(*)$, there are directed (x, u_k)-walks of every length $k - j + t\kappa(D_1)$ with $t \geqslant N$ for each $k = 1, \ldots, \kappa(D_1)$. Thus v is an m-step prey of x for each $m \geqslant N\kappa(D_1) + 1$. Since x is arbitrarily chosen, $C(D^m)$ is the union of two complete graphs whose vertex sets are $V(D_1)$ and $\{v\}$ for each $m \geqslant N\kappa(D_1) + 1$ and so $K_{|V(D_1)|} \cup \{v\}$ is the limit of $\{C(D^m)\}_{m=1}^{\infty}$.

We show the 'only if' part. Suppose that $\left|\left\{i \mid U_i^{(1)} \cap N_D^-(v) \neq \emptyset\right\}\right| = t$ for some $t \in \{2, \ldots, \kappa(D_1) - 1\}$. Then there exists j such that v has in-neighbors in $U_j^{(1)}$ and $U_{j+r}^{(1)}$ but no in-neighbor in $U_{j+1}^{(1)}$ where $r \in \{2, 3, \ldots, \kappa(D_1) - 1\}$. Take a vertex u in $U_j^{(1)}$ and a vertex w in $U_{j+r}^{(1)}$. Then, by $(*)$, v is a $(t\kappa(D_1) + 1)$-step common prey of u and w for each $t \geqslant N$. Thus u and w are adjacent in $C(D^{t\kappa(D_1)+1})$ for each $t \geqslant N$. However, u and w are not adjacent in $C(D^{t\kappa(D_1)+2})$ for any positive integer $t \geqslant N$. To show it, note that v is the only possible m-step common prey of u and w for any integer m and so suppose that v is a $(t\kappa(D_1) + 2)$-step prey of u for some positive integer $t \geqslant N$. We will reach a contradiction. By our assumption, there is a directed (u, v)-walk of length $t\kappa(D_1) + 2$ in D. The vertex immediately followed by v on the walk must belong to $U_{j+1}^{(1)}$, which contradicts our assumption that v does not have an in-neighbor in $U_{j+1}^{(1)}$. Thus v cannot be a $(t\kappa(D_1) + 2)$-step prey of u for any positive integer $t \geqslant N$. Therefore u and w are not adjacent in $C(D^{t\kappa(D_1)+2})$ for any positive integer $t \geqslant N$. Hence we can conclude that $\{C(D^m)\}_{m=1}^{\infty}$ does not converge. $\square$

If D_2 is nontrivial, then $\{C(D^m)\}_{m=1}^{\infty}$ converges by Theorem 2.1. Thus we have completely characterized a digraph D with exactly two strong components for which $\{C(D^m)\}_{m=1}^{\infty}$ converges.

Theorem 3.5. *Let D be a weakly connected digraph with exactly two strong components D_1 and D_2 and without arc from D_2 to D_1. Then $\{C(D^m)\}_{m=1}^{\infty}$ converges if and only if D satisfies one of the following:*

(i) D_2 is nontrivial.
(ii) D_2 is trivial and, for the vertex v of D_2, $\left|\left\{i \mid U_i^{(1)} \cap N_D^-(v) \neq \emptyset\right\}\right| = 1$ or $\kappa(D_1)$.

From the proof of Proposition 3.4 and Theorem 3.5, we obtain the following:

Corollary 3.6. *Let A be a Boolean $(0, 1)$-matrix in the following form:*

$$\begin{bmatrix} A_1 & F \\ O & A_2 \end{bmatrix}$$

where O is a zero matrix, F is a nonzero matrix, and A_1 and A_2 are irreducible. Then $\{\Gamma(A^m)\}_{m=1}^{\infty}$ converges if and only if one of the following holds:

(i) A_2 has order at least 2.
(ii) A_2 has order 1 and there exists a permutation matrix P such that

$$PAP^T = \left(\begin{array}{cccccccc|c} A_{12} & * & O & O & \cdots & O & O & & \\ O & A_{23} & * & O & \cdots & O & O & & F' \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & & \\ O & O & O & O & \cdots & A_{\kappa(D_1)-2,\kappa(D_1)-1} & * & & \\ * & O & O & O & \cdots & O & A_{\kappa(D_1)-1,\kappa(D_1)} & & \\ \hline & & & & O & & & & A_2 \end{array}\right)$$

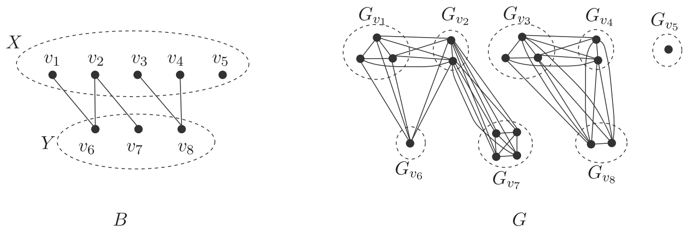

Fig. 2. A bipartite graph B and an expansion G of B.

so that the rows containing nonzero elements of F' intersect exactly one of $A_{12}, A_{23}, \ldots, A_{\kappa(D_1)-1,\kappa(D_1)}$ or each of $A_{12}, A_{23}, \ldots, A_{\kappa(D_1)-1,\kappa(D_1)}$. Moreover, the limit of $\{\Gamma(A^m)\}_{m=1}^{\infty}$ is a JBD matrix.

We now examine the structure of $\{\Gamma(A^m)\}_{m=1}^{\infty}$ when it converges where A is a Boolean $(0, 1)$-matrix given in (4) where F is a nonzero matrix. If A_2 is a trivial matrix, then we presented the limit of $\{\Gamma(A^m)\}_{m=1}^{\infty}$ in Corollary 3.6. Thus in the following, we find the limit when A_2 is nontrivial. To do so, we find the limit of $\{C(D^m)\}_{m=1}^{\infty}$ where D is the digraph of A and need the following useful notions.

Given a bipartite $B = (X, Y)$, we construct a supergraph of B as follows. We write each edge of B in the arc form (x, y) to make clear that $x \in X$ and $y \in Y$. Then we replace each vertex v with a complete graph G_v (of any size) so that G_v and G_w are vertex-disjoint if $v \neq w$, and join each vertex of G_x and each vertex of G_y whenever either (x, y) is an edge of B or there exists $z \in Y$ such that (x, z) and (y, z) are edges of B. We say that the resulting graph D is an *expansion of B*. (See Fig. 2 for an illustration.)

Lemma 3.7. *Let G be an expansion of some bipartite graph $B = (X, Y)$. Then G has only complete components if and only if for each vertex $x \in X$, the degree of x is at most one in B.*

Proof. We show the 'if' part by contradiction. Suppose that there exist vertices x, y, z such that xy and xz are edges of G but y is not adjacent to z in G. Let G_u, G_v, and G_w be the complete graphs replacing vertices u, v, and w of B containing x, y, z, respectively. By definition, u, v, and w are distinct. Since y and z are not adjacent while x is adjacent to both y and z, it is true that $u \in X$. Then, since B is bipartite, v and w belong to Y. Now, by definition, u is adjacent to v and w and we reach a contradiction.

To show the 'only if' part, suppose that G has only complete components and there exists a vertex $u \in X$ which has two neighbors v, w in Y. Then, by definition, no vertex of G_v is joined to any vertex of G_w. Take a vertex $x \in G_u$, a vertex $y \in G_v$ and a vertex $z \in G_w$. Then, by definition, x is adjacent to y and z in G and so x, y, z belong to the same component. Since G has only complete components by our assumption, y and z are adjacent in G, a contradiction. $\square$

Definition 3.8. We take a weakly connected digraph D with exactly two strong components D_1 and D_2 where D_2 is nontrivial. Let $I(D) = \{(k, l) \mid (x, y) \in A(D) \text{ for some } x \in U_k^{(1)}, y \in U_l^{(2)}\}$. Let $B_D = (\mathbb{Z}_{\kappa(D_1)}, \mathbb{Z}_{\kappa(D_2)})$ be the bipartite graph defined as follows. If D_1 is nontrivial, then B_D has an edge (i, j) if and only if $i \equiv k + 1 + p \pmod{\kappa(D_1)}$ and $j \equiv l + p \pmod{\kappa(D_2)}$ for some $(k, l) \in I(D)$ and some integer p. If D_1 is trivial, then B_D has an edge (i, j) if and only if $j \equiv l - 1 \pmod{\kappa(D_2)}$ for some $(1, l) \in I(D)$, which is obtained by substituting $p = -1$ and $k(D_1) = 1$ in the nontrivial case.

We note that when we consider an edge (x, y) of B_D, the first component and the second component are reduced modulo $\kappa(D_1)$ and $\kappa(D_2)$, respectively. Then following is true.

Lemma 3.9. *Let D be a weakly connected digraph with exactly two strong components D_1 and D_2 where D_2 is nontrivial. Then (i, j) is an edge of B_D if and only if there exists a (u, v)-walk of length $2s\kappa(D_1)\kappa(D_2)$ for some vertices $u \in U_i^{(1)}$ and $v \in U_j^{(2)}$ and for some positive integer s.*

Proof. For simplicity, we denote $\kappa(D_1)\kappa(D_2)$ by λ. To show the 'only if' part, suppose that (i, j) is an edge of B_D. By definition, for some integer p and for some $(k, l) \in I(D)$,

$$i \equiv k + 1 + p \pmod{\kappa(D_1)}, \qquad j \equiv l + p \pmod{\kappa(D_2)}.$$

We define nonnegative integers p_1 and p_2 as follows. There exists a positive integer s such that both $s\lambda - p - 1$ and $s\lambda + p$ are positive integers. If D_1 is trivial, then let $p_1 = 0$ and $p_2 = 2s\lambda + p$ (note that $p = -1$). If D_1 is nontrivial, then let $p_1 = s\lambda - p - 1$ and $p_2 = s\lambda + p$.

Since $(k, l) \in I(D)$, there exist vertices $x \in U_k^{(1)}$ and $y \in U_l^{(2)}$ such that (x, y) is an arc of D. If D_1 is trivial, then let $u = x$ and then $Q := u$ is a (u, x)-walk of length $p_1 = 0$. Suppose that D_1 is nontrivial. Since D_1 is a nontrivial strongly connected digraph, any vertex in D_1 has an in-neighbor in D_1 and so there is a directed (u, x)-walk Q of length p_1 for some $u \in V(D_1)$. Since $x \in U_k^{(1)}$, it is true that $u \in U_{k-p_1}^{(1)}$. However,

$$k - p_1 \equiv (i - p - 1) - (s\lambda - p - 1) \equiv i \pmod{\kappa(D_1)}.$$

Therefore $u \in U_i^{(1)}$.

Since D_2 is nontrivial and strongly connected D_2 has a directed (y, v)-walk R of length p_2 for some $v \in V(D_2)$. Therefore $y \in U_{l+p_2}^{(2)}$. However,

$$l + p_2 \equiv (j - p) + (ks\lambda + p) \equiv j \pmod{\kappa(D_2)},$$

where $k = 1$ if D_1 is nontrivial and $k = 2$ if D_1 is trivial. Thus $v \in U_j^{(2)}$.

Furthermore $Q + (x, y) + R$ has the length $p_1 + p_2 + 1 = 2s\lambda$.

To show the 'if' part, suppose that there exists a directed (u, v)-walk Q of length $2s\lambda$ for some vertices $u \in U_i^{(1)}$ and $v \in U_j^{(2)}$ and for some positive integer s. The walk Q contains a unique arc (w, z) such that $w \in V(D_1)$ and $z \in V(D_2)$ since there is no arc from a vertex in D_2 to a vertex in D_1. Then $w \in U_r^{(1)}$ and $z \in U_s^{(2)}$ for some $r \in \{1, 2, \ldots, \kappa(D_1)\}$ and some $s \in \{1, 2, \ldots, \kappa(D_2)\}$. By definition, $(r, s) \in I(D)$. Let ℓ_1 and ℓ_2 be the lengths of (u, w)-section of Q and (z, v)-section of Q, respectively. Then

$$i + \ell_1 \equiv r \pmod{\kappa(D_1)}, \qquad j - \ell_2 \equiv s \pmod{\kappa(D_2)}.$$

Since $\ell_1 + \ell_2 + 1 = 2s\lambda$, the first congruence relation is equivalent to $i + (2s\lambda - \ell_2 - 1) \equiv r \pmod{\kappa(D_1)}$ and so $i - \ell_2 - 1 \equiv r \pmod{\kappa(D_1)}$. Therefore

$$i \equiv r + \ell_2 + 1 \pmod{\kappa(D_1)}, \quad j \equiv s + \ell_2 \pmod{\kappa(D_2)}.$$

By definition, (i, j) is an edge of B_D. $\square$

Now we are ready to present the limit of $\{\Gamma(A^m)\}_{m=1}^{\infty}$ if it exists for a matrix A given in (4) when F is a nonzero matrix and A_2 has order at least 2, that is, $D(A_2)$ is a nontrivial strong component of $D(A)$:

Theorem 3.10. *Let D be a weakly connected digraph with two strong components D_1 and D_2 such that no arc goes from D_2 to D_1, D_2 is nontrivial, and $\{C(D^m)\}_{m=1}^{\infty}$ converges. Then the limit of $\{C(D^m)\}_{m=1}^{\infty}$ is an expansion of the bipartite graph B_D defined in Definition 3.8.*

Proof. Let G be the limit of $\{C(D^m)\}_{m=1}^{\infty}$. By Theorem 3.2, complete graphs whose vertex sets are $U_1^{(1)}$, ..., $U_{\kappa(D_1)}^{(1)}$, $U_1^{(2)}$, ..., $U_{\kappa(D_2)}^{(2)}$, respectively, are subgraphs of G.

Since there is no arc from a vertex in D_2 to a vertex in D_1, for any positive integer m, an m-step common prey of two vertices of $V(D_2)$ is in D_2 and so the union of complete graphs whose vertex sets are $U_1^{(2)}, \ldots, U_{\kappa(D_2)}^{(2)}$, respectively, is an induced subgraph of G by Theorem 2.3. Therefore, to show that G is an expansion of B_D, it is sufficient to prove the following:

(i) $x \in U_i^{(1)}$ and $y \in U_j^{(2)}$ are adjacent in G if and only if (i, j) is an edge of B_D.

(ii) For distinct i and j, $x \in U_i^{(1)}$ and $y \in U_j^{(1)}$ are adjacent in G if and only if (i, h) and (j, h) are edges of B_D for some $h \in \mathbb{Z}_{\kappa(D_2)}$.

For simplicity, we denote $\kappa(D_1)\kappa(D_2)$ by λ.

To show (i), suppose that $x \in U_i^{(1)}$ and $y \in U_j^{(2)}$ are adjacent in G. Then there exist an integer s and a $2s\lambda$-step common prey z of x and y in D, which implies that there exists an (x, z)-walk Q of length $2s\lambda$ in D. On the other hand, since z is a $2s\lambda$-step prey of y, it is true that $z \in V(D_2)$. Furthermore, since $2s\lambda$ is a multiple of $\kappa(D_2)$, $z \in U_j^{(2)}$. By Lemma 3.9, (i, j) is an edge of B_D.

Suppose that (i, j) is an edge of B_D. Take vertices $x \in U_i^{(1)}$ and $y \in U_j^{(2)}$. By Lemma 3.9, there exists a (u, v)-walk Q of length $2s\lambda$ for some vertices $u \in U_i^{(1)}$ and $v \in U_j^{(2)}$ and some positive integer s. Since any directed (x, u)-walk belongs to D_1, the length of a directed (x, u)-walk is congruent to 0 modulo $\kappa(D_1)$. Then there exists a directed (x, u)-walk S of length $s'\lambda$ for some integer s' (If D_1 is trivial then $s' = 0$). Since D_2 is nontrivial and $\kappa(D_2)$ divides λ, by Lemma 3.3, for some integer N, there exist a directed (v, v)-walk T of length $N\lambda$, and a directed (y, v)-walk R of length $(2s + s' + N)\lambda$. Since SQT is a directed (x, v)-walk of length $(2s + s' + N)\lambda$, v is a $(2s + s' + N)\lambda$-step common prey of x and y in D, and so x and y are adjacent in G. Hence (i) holds.

Now we show that (ii) holds. Take distinct i and j in $\{1, \ldots, \kappa(D_1)\}$. To prove the 'if' part, suppose that (i, h) and (j, h) are edges of B_D for some $h \in \mathbb{Z}_{\kappa(D_2)}$. By Lemma 3.9, there exist a directed (u_1, v_1)-walk S_1 of length $2s_1\lambda$ and a directed (u_2, v_2)-walk S_2 of length $2s_2\lambda$ for some positive integers s_1 and s_2, and some vertices $u_1 \in U_i^{(1)}$, $u_2 \in U_j^{(1)}$, $v_1, v_2 \in U_h^{(2)}$. Take two vertices $x \in U_i^{(1)}$ and $y \in U_j^{(1)}$. Note that assumption $i \neq j$ implies that D_1 is nontrivial. Therefore both D_1 and D_2 are nontrivial, we may apply Lemma 3.3 to D_1 and D_2, respectively, to have integers N_1 and N_2 satisfying the following. Since v_1, v_2 belong to the same set of imprimitivity, there exist a directed (v_1, v_2)-walk T_1 of length $(2s_2+N_2)\lambda$ and a directed (v_2, v_2)-walk T_2 of length $(2s_1+N_2)\lambda$. Then S_1T_1 is a directed (u_1, v_2)-walk of length $(2s_1 + 2s_2 + N_2)\lambda$ and S_2T_2 is a directed (u_2, v_2)-walk of length $(2s_2 + 2s_1 + N_2)\lambda$. Take any integer $t \geqslant N_1$. Then, since x, u_1 belong to the same set of imprimitivity, there exists a directed (x, u_1)-walk Q_1 of length $t\lambda$. For the same reason, there exists a directed (y, u_2)-walk Q_2 of length $t\lambda$. Now $Q_1S_1T_1$ is a directed (x, v_2)-walk of length $(2s_1 + 2s_2 + N_2 + t)\lambda$, and $Q_2S_2T_2$ is a directed (y, v_2)-walk of length $(2s_1 + 2s_2 + N_2 + t)\lambda$. Thus v_2 is a $(2s_1 + 2s_2 + N_2 + t)\lambda$-step common prey of x and y, and hence x and y are adjacent in $C(D^{(2s_1+2s_2+N_2+t)\lambda})$. Since t is arbitrarily chosen integer greater than N_1, x and y are adjacent in G.

To show the 'only if' part, suppose that $x \in U_i^{(1)}$ and $y \in U_j^{(1)}$ are adjacent in G. Then they have a $2s\lambda$-step common prey z in $V(D_2)$ for some integer s, that is, there exist a directed (x, z)-walk and a directed (y, z)-walk of length $2s\lambda$. Since $z \in V(D_2)$, it holds that $z \in U_h^{(2)}$ for some $h \in \{1, 2, \ldots, \kappa(D_2)\}$. By Lemma 3.9, (i, h) and (j, h) are edges of B_D. $\square$

We can easily check that Theorem 3.10 is equivalent to the following:

Corollary 3.11. *Let $A \in \mathcal{B}_n$ be a matrix such that for a permutation matrix P of order n,*

$$PAP^T = \begin{bmatrix} A_1 & F \\ O & A_2 \end{bmatrix}$$

where O is a zero matrix,

$$A_1 = \begin{bmatrix} O & A_{12} & O & \cdots & O \\ O & O & A_{23} & \cdots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & O & \cdots & A_{\kappa(D_1)-1,\kappa(D_1)} \\ A_{\kappa(D_1)1} & O & O & \cdots & O \end{bmatrix}, \quad A_2 = \begin{bmatrix} O & B_{12} & O & \cdots & O \\ O & O & B_{23} & \cdots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & O & \cdots & B_{\kappa(D_2)-1,\kappa(D_2)} \\ B_{\kappa(D_2)1} & O & O & \cdots & O \end{bmatrix},$$

and A_2 has order at least two, and F is a nonzero matrix,

$$F = \begin{bmatrix} F_{11} & F_{12} & F_{13} & \cdots & F_{1\kappa(D_2)} \\ F_{21} & F_{22} & F_{23} & \cdots & F_{2\kappa(D_2)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ F_{\kappa(D_1)-1,1} & F_{\kappa(D_1)-1,2} & F_{\kappa(D_1)-1,3} & \cdots & F_{\kappa(D_1)-1,\kappa(D_2)} \\ F_{\kappa(D_1)1} & F_{\kappa(D_1)2} & F_{\kappa(D_1)3} & \cdots & F_{\kappa(D_1)\kappa(D_2)} \end{bmatrix}.$$

Then $\{\Gamma(A^m)\}_{m=1}^{\infty}$ converges to a matrix A' such that

$$PA'P^T = \begin{bmatrix} C_1 & F' \\ F'^T & C_2 \end{bmatrix}$$

where

$$C_1 = \begin{bmatrix} J & C_{12} & C_{13} & \cdots & C_{1\kappa(D_1)} \\ C_{21} & J & C_{23} & \cdots & C_{2\kappa(D_1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_{\kappa(D_1)-1,1} & C_{\kappa(D_1)-1,2} & C_{\kappa(D_1)-1,3} & \cdots & C_{\kappa(D_1)-1,\kappa(D_1)} \\ C_{\kappa(D_1)1} & C_{\kappa(D_1)2} & C_{\kappa(D_1)3} & \cdots & J \end{bmatrix}; \quad C_2 = \begin{bmatrix} J & O & O & \cdots & O \\ O & J & O & \cdots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & O & \cdots & O \\ O & O & O & \cdots & J \end{bmatrix};$$

$$F' = \begin{bmatrix} F'_{11} & F'_{12} & F'_{13} & \cdots & F'_{1\kappa(D_2)} \\ F'_{21} & F'_{22} & F'_{23} & \cdots & F'_{2\kappa(D_2)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ F'_{\kappa(D_1)-1,1} & F'_{\kappa(D_1)-1,2} & F'_{\kappa(D_1)-1,3} & \cdots & F'_{\kappa(D_1)-1,\kappa(D_2)} \\ F'_{\kappa(D_1)1} & F'_{\kappa(D_1)2} & F'_{\kappa(D_1)3} & \cdots & F'_{\kappa(D_1)\kappa(D_2)} \end{bmatrix};$$

J represents a matrix of an appropriate size with all the elements 1; $C_{ij} = C_{ji} = J$ if $F'_{ik} = F'_{jk} = J$ for some $k \in \{1, \ldots, \kappa(D_2)\}$ and $C_{ij} = C_{ji} = O$ otherwise; $F'_{ij} = J$ if one of the following holds:

- *A_1 has order at least two and $F_{k,l} \neq O$ for some integers k, l satisfying $i \equiv k + p + 1 \pmod{\kappa(D_2)}$ and $j \equiv l + p \pmod{\kappa(D_2)}$ for some integer p,*
- *A_1 has order one and $F_{1l} \neq O$ for some integer l such that $j \equiv l - 1 \pmod{\kappa(D_2)}$,*

and $F'_{ij} = O$ otherwise.

Let A be a matrix given in (4) where F is nonzero and D be the digraph of A. If D_1 is trivial, D_2 is nontrivial and B_D has at least two edges, then any expansion of B_D cannot be the union of complete subgraphs and so the limit of $\{C(D^m)\}_{m=1}^{\infty}$ is not the union of complete subgraphs by Theorem 3.10, that is, the limit of $\{\Gamma(A^m)\}_{m=1}^{\infty}$ cannot be a JBD matrix. If D_2 trivial and $\{C(D^m)\}_{m=1}^{\infty}$ converges, then the limit of $\{C(D^m)\}_{m=1}^{\infty}$ is always the union of complete subgraphs by Proposition 3.4, that is, the limit of $\{\Gamma(A^m)\}_{m=1}^{\infty}$ is a JBD matrix. In the following, we characterize a matrix A given in (4) for which the limit graph of $\{\Gamma(A^m)\}_{m=1}^{\infty}$ is a JBD matrix when F is a nonzero matrix and both A_1 and A_2 are nontrivial, that is, we characterize a digraph D with exactly two strong components both of which are nontrivial and for which the limit graph of $\{C(D^m)\}_{m=1}^{\infty}$ has only complete components.

Theorem 3.12. *Let D be a weakly connected digraph with exactly two strong components D_1 and D_2 both of which are nontrivial and without arc from D_2 to D_1. Suppose that $\{C(D^m)\}_{m=1}^{\infty}$ converges to a graph G. Then G is the union of complete subgraphs if and only if $\kappa(D_2)$ divides $\kappa(D_1)$ and $i - j \equiv i' - j' \pmod{\kappa(D_2)}$ for any (i, j), $(i', j') \in I(D)$.*

Proof. As we have shown in the proof of Theorem 3.10, G is an expansion of the bipartite graph B_D defined in Definition 3.8. For convenience, let $B_D = (X, Y)$. Suppose that $\kappa(D_2)|\kappa(D_1)$ and $j-i \equiv j'-i' \pmod{\kappa(D_2)}$ for any (i, j), $(i', j') \in I(D)$. Take a vertex $a \in X$. If a has no neighbor in B_D, then its degree is zero. Suppose that a has a neighbor in B_D. Let (a, b) and (a, c) are edges of B_D. Then by definition, for some (i, j), $(i', j') \in I(D)$, for some integers l, l',

$$\begin{aligned} &a \equiv i + l + 1 \pmod{\kappa(D_1)}, \quad b \equiv j + l \pmod{\kappa(D_2)}, \\ &a \equiv i' + l' + 1 \pmod{\kappa(D_1)}, \quad c \equiv j' + l' \pmod{\kappa(D_2)}. \end{aligned}$$

Since $\kappa(D_2)|\kappa(D_1)$,

$$a \equiv i + l + 1 \equiv i' + l' + 1 \pmod{\kappa(D_2)},$$

and so $l - l' \equiv i' - i \pmod{\kappa(D_2)}$. Therefore

$$b - c \equiv (j + l) - (j' + l') \equiv (j - j') + (l - l') \equiv (j - j') + (i' - i) \equiv 0 \pmod{\kappa(D_2)}.$$

Therefore, the vertex a has only one neighbor in B_D. Hence, by Lemma 3.7, G is the union of complete subgraphs.

Now suppose that G is the union of complete subgraphs. Then, by Lemma 3.7, the degree of each vertex in X is at most one in B_D. Since we have assumed that the underlying graph of D is connected at the beginning of this section, B_D has an edge and so there exists a vertex $a \in X$ such that the degree of a is one. Then (a, b) is an edge of B_D and so for some $(i, j) \in I(D)$ and some integer l,

$$a \equiv i + l + 1 \pmod{\kappa(D_1)}, \qquad b \equiv j + l \pmod{\kappa(D_2)}.$$

Since $(i, j) \in I(D)$, it is true that $(i + l + \kappa(D_1) + 1, j + l + \kappa(D_1))$ is an edge of B_D by the definition of B_D. Since $i + l + \kappa(D_1) + 1 \equiv a \pmod{\kappa(D_1)}$, it holds that $(a, j + l + \kappa(D_1))$ is an edge of B_D. Since the degree of a is one, b is the unique neighbor of a in B_D and so $j + l + \kappa(D_1) \equiv b \pmod{\kappa(D_2)}$. Thus $j + l + \kappa(D_1) \equiv j + l \pmod{\kappa(D_2)}$ and so $\kappa(D_1) \equiv 0 \pmod{\kappa(D_2)}$. Therefore $\kappa(D_2)|\kappa(D_1)$.

Take (i, j), $(i', j') \in I(D)$. Without loss of generality, we may assume that $i' > i$. Since $(i, j) \in I(D)$, by the definition of B_D, $(i + (i' - i) + 1, j + (i' - i))$ is an edge of B_D and so is $(i' + 1, j + i' - i)$. In addition, $(i' + 1, j')$ is an edge of B_D as $(i', j') \in I(D)$. Therefore both $(i' + 1, j + i' - i)$ and $(i' + 1, j')$ are edges of B_D. Since each vertex in X of B_D has degree at most one, $j + (i' - i) \equiv j' \pmod{\kappa(D_2)}$. Thus $i - j \equiv i' - j' \pmod{\kappa(D_2)}$. $\square$

As a corollary of Theorem 3.12, we obtain the following:

Corollary 3.13. *Let $A \in \mathcal{B}_n$ be a matrix such that for a permutation matrix P of order n,*

$$PAP^T = \begin{bmatrix} A_1 & F \\ O & A_2 \end{bmatrix}$$

where

$$A_1 = \begin{bmatrix} O & A_{12} & O & \cdots & O \\ O & O & A_{23} & \cdots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & O & \cdots & A_{\kappa(D_1)-1,\kappa(D_1)} \\ A_{\kappa(D_1)1} & O & O & \cdots & O \end{bmatrix}, \quad A_2 = \begin{bmatrix} O & B_{12} & O & \cdots & O \\ O & O & B_{23} & \cdots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & O & \cdots & B_{\kappa(D_2)-1,\kappa(D_2)} \\ B_{\kappa(D_2)1} & O & O & \cdots & O \end{bmatrix},$$

both A_1 and A_2 have order at least two, O is a zero matrix, and F is a nonzero matrix,

$$F = \begin{bmatrix} F_{11} & F_{12} & F_{13} & \cdots & F_{1\kappa(D_2)} \\ F_{21} & F_{22} & F_{23} & \cdots & F_{2\kappa(D_2)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ F_{\kappa(D_1)-1,1} & F_{\kappa(D_1)-1,2} & F_{\kappa(D_1)-1,3} & \cdots & F_{\kappa(D_1)-1,\kappa(D_2)} \\ F_{\kappa(D_1)1} & F_{\kappa(D_1)2} & F_{\kappa(D_1)3} & \cdots & F_{\kappa(D_1)\kappa(D_2)} \end{bmatrix}.$$

Suppose that $\{\Gamma(A^m)\}_{m=1}^{\infty}$ converges to a matrix A'. Then A' is a JBD matrix if and only if $\kappa(D_2)$ divides $\kappa(D_1)$ and $i - j \equiv i' - j' \pmod{\kappa(D_2)}$ whenever $F_{i,j} \neq O$ and $F_{i',j'} \neq O$.

4. Concluding remarks

In this paper, we investigated the convergence and the limit of the matrix sequence $\{\Gamma(A^m)\}_{m=1}^{\infty}$ for a matrix A in $\mathcal{B}_n$ whose digraph D has at most two strong components and, among such matrices, characterized a matrix A for which the limit of $\{\Gamma(A^m)\}_{m=1}^{\infty}$ is a JBD matrix. We would like to see if our results can be generalized for an arbitrary matrix in $\mathcal{B}_n$. When a digraph D has quite many strong components, vertices in the strong component which has only outgoing arcs in the condensation of D have much more choices for prey and so the characterization of its limit, if it exists, appears to be more difficult.

We mentioned earlier that studying the matrix sequence $\{\Gamma(A^m)\}_{m=1}^{\infty}$ for a matrix A in $\mathcal{B}_n$ is equivalent to studying the graph sequence $\{C(D^m)\}_{m=1}^{\infty}$ and that $\{C(D^m)\}_{m=1}^{\infty}$ is actually the sequence of m-step competition graphs of D. In this context, we propose to investigate the graph sequence obtained by other variants of competition graph (see [4,9,15]).

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